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BOUNDARY ELEMENT ANALYSIS OF SHALLOW SHELLS INVOLVING SHEAR DEFORMATION

PIN LU and MAOGUANG HUANG
Department of Modern Mechanics, University of Science and Technology of China, Hefei, Anhui 230026, P.R.C.

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Abstract—Based on the fundamental solutions previously obtained by the authors, a boundary integral approach to shallow shells involving shear deformation is presented, and related formulae are derived. This provides a new method for solving this kind of shell problem.

INTRODUCTION

In recent years, considerable progress has been made in the application of the boundary element method to plate and shell bending as well as to other engineering problems. However, for the kind of shallow shells involving shear deformation and with arbitrary quadratic middle surfaces, the corresponding BEM analysis has not been performed before because it is difficult and complicated to find the fundamental solutions to this kind of shell. It is known that the shallow shell theory considering shear deformation has many applications in engineering problems, such as stress concentration, high order vibration and thick shell bending, etc. Therefore, it would be significant both in theory and application to develop a BEM approach to solving the problems for this kind of shell.

For this reason, we derived, as the most important step of the BEM application, the fundamental solutions of this shell theory by using Hörmander's operator and plane-wave decomposition methods (Lu and Huang, 1991) and corresponding computational methods and treatments were also discussed in detail. In the present paper, BEM formulations for this shell model have been established with the kernels obtained by the authors, and some related problems have been studied. All of these investigations have extended the BEM approach to this kind of shell problem. In addition, a program has been written according to the formulations obtained, and some numerical examples are given at the end of the paper.

A BRIEF REVIEW OF THE BASIC EQUATIONS

Consider a shallow shell involving shear deformation and with a quadratic middle surface given by

$$z = -\tfrac{1}{2}(k_1x^2+k_2y^2) \tag{1}$$

where k_1 and k_2 are the principal curvatures of the shell in the x- and y-directions respectively. The equilibrium equations of the shell are (Sih, 1977):

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y} = p_x,$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y} = p_y,$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}-(k_1N_x+k_2N_y)+p_z = 0,$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=m_x,$$
$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=m_y \tag{2}$$

and the corresponding stress–displacement relations are:

$$N_x=B\left[\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+(k_1+vk_2)w\right],$$
$$N_y=B\left[\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial x}+(k_2+vk_1)w\right],$$
$$N_{xy}=\tfrac{1}{2}(1-v)B\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$
$$Q_x=C\left(\frac{\partial w}{\partial x}+\psi_x\right),$$
$$Q_y=C\left(\frac{\partial w}{\partial y}+\psi_y\right),$$
$$M_x=D\left(\frac{\partial \psi_x}{\partial x}+v\frac{\partial \psi_y}{\partial y}\right),$$
$$M_y=D\left(\frac{\partial \psi_y}{\partial y}+v\frac{\partial \psi_x}{\partial x}\right),$$
$$M_{xy}=\tfrac{1}{2}(1-v)D\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right) \tag{3}$$

where u, v, w, ψ_x and ψ_y are five independent displacements p_x, p_y, p_z, m_x and m_y are generalized distributive loads applied in different directions of the shell, B, C and D are tension, shear and bending stiffness respectively and v is Poisson's ratio.

Substituting eqn (3) into eqn (2), we can obtain the equilibrium equations expressed in terms of displacements:

$$[\mathbb{L}]\{\mathbb{U}\}=\{\mathbb{P}\} \tag{4}$$

where

$$\{\mathbb{U}\}=[u,v,w,\psi_x,\psi_y]^{\mathrm{T}},$$
$$\{\mathbb{P}\}=[p_x,p_y,p_z,m_x,m_y]^{\mathrm{T}} \tag{5}$$

and $[\mathbb{L}]$ is a symmetrical differential operator matrix of the order 5×5. [See Lu and Huang, (1991).]

It is very difficult to solve the partial differential equations (4) directly. However, the decomposed forms of the equations have been obtained by the authors:

$$L\{\Phi^*\}=\frac{1}{D}\{\mathbb{P}\} \tag{6}$$

where

$$\{\Phi^*\} = [\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5]^T \tag{7}$$

are five displacement functions, L is a 10th-order partial differential operator, and is given in the Appendix. Therefore, the generalized displacements and stress resultants can be obtained respectively from the relations

$$\{\mathbb{U}\} = [\mathbb{F}]\{\Phi^*\} \tag{8}$$

$$\{\mathbb{T}\} = [\mathbb{R}]\{\Phi^*\} \tag{9}$$

where

$$\{\mathbb{T}\} = [N_x, N_y, N_{xy}, Q_x, Q_y, M_y, M_x, M_{xy}]^T \tag{10}$$

and $[\mathbb{F}]$ and $[\mathbb{R}]$ are also differential operator matrices, which are given in the Appendix. Let $\Phi(x, y)$ be the fundamental solution of the differential operator L, i.e.

$$L[\Phi(x, y)] = \delta(x, y) \tag{11}$$

where $\delta(x, y)$ is the Dirac δ-function. Then the displacement function satisfied by eqn (6) can be expressed as

$$\Phi_j(x, y) = \iint \Phi(x-\zeta, y-\eta)\frac{p_j}{D}\,\mathrm{d}\zeta\,\mathrm{d}\eta \quad (j = 1, 2, \ldots, 5) \tag{12}$$

once $\Phi_j(x, y)$ has been evaluated, the displacements and stress resultants can be obtained from eqns (8) and (9), respectively. Now the problems of solving eqn (6) have been reduced to finding the fundamental solution $\Phi(x, y)$ which satisfied eqn (11). This solution has been obtained by the authors:

$$\Phi(x, y) = \int_0^{2\pi} \phi(\rho)\ \mathrm{d}\theta \tag{13}$$

and

$$\phi(\rho) = \sum_{j=1}^{3} \Lambda_j \sum_{m=1}^{\infty} \frac{(r_j\rho)^{2m+2}}{(2m+2)!}[\gamma + \ln|r_j\rho| + i\beta_j - \psi(2m+3)] \tag{14}$$

where

$$\rho = x\cos\theta + y\sin\theta,$$

$$r_1 = a_1, \quad r_2 = \eta_1 + i\eta_2, \quad r_3 = \eta_1 - i\eta_2,$$

$$\Lambda_1 = \frac{1}{4\pi^2 r_1^4 (r_1^2 - r_2^2)(r_1^2 - r_3^2)},$$

$$\Lambda_2 = \frac{1}{4\pi^2 r_2^4 (r_2^2 - r_1^2)(r_2^2 - r_3^2)},$$

$$\Lambda_3 = \frac{1}{4\pi^2 r_3^4 (r_3^2 - r_1^2)(r_3^2 - r_2^2)},$$

$$\beta_1 = 0, \quad \beta_2 = \arctan|\eta_2/\eta_1|, \quad \beta_3 = -\beta_2,$$

$$\psi(m+1) = \sum_{s=1}^{m} \frac{1}{s}, \quad \psi(1) = 0, \quad \gamma \cong 0.51721 \text{ (Euler's constant)},$$

$$a_1 = \left(\frac{2}{1-\nu}\frac{C}{D}\right)^{1/2}, \quad a_2 = (k_1 \sin^2\theta + k_2 \cos^2\theta)(1-\nu^2)\frac{B}{C}, \quad a_3 = a_2\frac{C}{D},$$

$$\eta_1 = (\sqrt{a_3/4} + a_2/4)^{1/2}, \quad \eta_2 = (\sqrt{a_3/4} - a_2/4)^{1/2} \tag{15}$$

and corresponding methods and strategies for the numerical computation of the fundamental solutions have also been discussed in detail in the authors' previous paper.

BEM FORMULATIONS

Boundary integral equations for elasticity problems can be established using Betti's reciprocal theorem or weighted residual method. There are five independent displacements for the theory of shallow shells involving shear deformation. The related boundary integral equations can be written as

$$C_{ij}(P)u_j(P) = \int_\Gamma [T_{ij}(P,Q)u_j(Q) - U_{ij}(P,Q)t_j(Q)]\,\mathrm{d}\Gamma_Q + \iint_\Omega U_{ij}(P,Q)p_j(Q)\,\mathrm{d}\Omega_Q \quad (i,j = 1,\ldots,5) \tag{16}$$

and it can be rewritten in matrix form as:

$$[\mathbb{C}]\{\mathbb{U}\} = \int_\Gamma ([\mathbb{T}]^*\{\mathbb{U}\} - \{\mathbb{U}\}^*\{\mathbb{T}\})\,\mathrm{d}\Gamma_Q + \iint_\Omega [\mathbb{U}]^*\{\mathbb{P}\}\,\mathrm{d}\Omega_Q \tag{17}$$

where

$$\begin{aligned} \{\mathbb{U}\} &= [u, v, w, \psi_x, \psi_y]^T, \\ \{\mathbb{T}\} &= [N_1, N_2, Q_1, M_1, M_2]^T, \\ \{\mathbb{P}\} &= [p_x, p_y, p_z, m_x, m_y]^T \end{aligned} \tag{18}$$

and

$$\begin{aligned} N_1 &= N_x n_x + N_{xy} n_y, \quad N_2 = N_{xy} n_x + N_y n_y, \\ M_1 &= M_x n_x + M_{xy} n_y, \quad M_2 = M_{xy} n_x + M_y n_y, \\ Q_1 &= Q_x n_x + Q_y n_y \end{aligned} \tag{19}$$

represent the stress resultants at the boundary of the shell in the x- and y-directions respectively. $U_{ij}(P,Q)$ and $T_{ij}(P,Q)$ represent the displacements and tractions in the i-direction at point Q corresponding to a unit point force acting in the j-direction applied at point P, and can be obtained using the following methods.

It can be seen from previous discussions that a set of corresponding displacement functions would be produced for each kind of unit force applied at point P. Therefore, five groups of displacement functions $\{\Phi^*\}_j$, $(j = 1,\ldots,5)$ can be obtained in this way, and according to eqn (12) the functions in the jth group are

$$\Phi_{jk} = \frac{\Phi}{D}\delta_{jk} \quad (k = 1,\ldots,5). \tag{20}$$

Furthermore, the following representations can be obtained according to eqn (8)

$$U_{ij}(P,Q)=\frac{1}{D}F_{ij}(\Phi),\quad (i,j=1,\ldots,5) \tag{21}$$

where differential operators F_{ij} are given in the Appendix, and Φ is the fundamental solution given in eqn (13). In addition, the expressions for the fundamental traction $T_{ij}(P,Q)$ can be obtained from eqns (19) and (9):

$$T_{1j}(P,Q)=\frac{1}{D}[n_xR_{1j}(\Phi)+n_yR_{3j}(\Phi)],$$

$$T_{2j}(P,Q)=\frac{1}{D}[n_xR_{3j}(\Phi)+n_yR_{2j}(\Phi)],$$

$$T_{3j}(P,Q)=\frac{1}{D}[n_xR_{4j}(\Phi)+n_yR_{5j}(\Phi)],$$

$$T_{4j}(P,Q)=\frac{1}{D}[n_xR_{6j}(\Phi)+n_yR_{8j}(\Phi)],$$

$$T_{5j}(P,Q)=\frac{1}{D}[n_xR_{8j}(\Phi)+n_yR_{7j}(\Phi)] \tag{22}$$

and differential operators R_{ij} are also listed in the Appendix, and the computations of $F_{ij}(\Phi)$ and $R_{ij}(\Phi)$ have been discussed in detail by the authors.

The coefficient $C_{ij}(p)$ in matrix [C] is determined by the position of source point P. If point P is in the domain, $C_{ii}=1$, $(i=1,2,4,5)$, $C_{33}=-1$, $C_{ij}=0$, $(i\neq j)$, and if P is on a smooth boundary, $C_{ii}=\frac{1}{2}$, $(i=1,2,4,5)$, $C_{33}=-\frac{1}{2}$, $C_{ij}=0$, $(i\neq j)$, when this point is on a non-smooth boundary, as shown in Fig. 1, it can be written as follows (Brebbia *et al.*, 1984):

$$C_{ij}=\delta_{ij}-\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}T_{ij}(P,Q)\,\mathrm{d}\Gamma_Q.$$

Substitute eqn (22) into the above expressions and after proper integrations and simplifications, the following relations can be obtained:

$$[\mathrm{C}]=\frac{1}{2\pi}\begin{bmatrix} c_2 & 0 & 0\\ 0 & -\pi+\theta_2-\theta_1 & 0\\ 0 & 0 & c_2\end{bmatrix} \tag{23}$$

where c_2 is a submatrix of order 2×2:

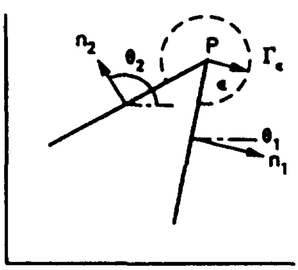

Fig. 1. Corner point.

$$c_2 = \begin{bmatrix} \pi+\theta_1-\theta_2-\frac{1+\nu}{4}[\sin(2\theta_1)-\sin(2\theta_2)] & \frac{1+\nu}{4}[\cos(2\theta_1)-\cos(2\theta_2)] \\ \frac{1+\nu}{4}[\cos(2\theta_1)-\cos(2\theta_2)] & \pi+\theta_1-\theta_2+\frac{1+\nu}{4}[\sin(2\theta_1)-\sin(2\theta_2)] \end{bmatrix} \tag{24}$$

where the meanings of θ_1 and θ_2 are shown in Fig. 1.

Treatments of body forces in boundary integral equations have been discussed in detail by Danson [see Brebbia (1984)]. For uniform normal distributed forces acting on shell surfaces, the domain integrals can be easily transformed into boundary integrals. In this case, the integrals corresponding to body forces in eqn (16) can be written as

$$B_i = p_z \iint_\Omega U_{i3}(P,Q)\,\mathrm{d}\Omega_Q = \frac{p_z}{D}\iint_\Omega F_{i3}(\Phi)\,\mathrm{d}\Omega_Q, \quad (i = 1,\ldots,5). \tag{25}$$

By means of Green's indentity and the formulations in the Appendix, the above expressions can be simplified further:

$$\begin{aligned} B_1 &= \frac{p_z}{C}\int_\Gamma n_x \tilde{L}_1\tilde{L}_2\tilde{L}'_8(\Phi)\,\mathrm{d}\Gamma_Q, \\ B_2 &= \frac{p_z}{C}\int_\Gamma n_y \tilde{L}_1\tilde{L}_2\tilde{L}'_9(\Phi)\,\mathrm{d}\Gamma_Q, \\ B_3 &= \frac{p_z}{C}\int_\Gamma \frac{\partial}{\partial n_Q}[\nabla^2\tilde{L}_1\tilde{L}_2(\Phi)]\,\mathrm{d}\Gamma_Q, \\ B_4 &= \frac{p_z}{C}\int_\Gamma \frac{\partial}{\partial n_Q}[\nabla^2\tilde{L}_1 D_1(\Phi)]\,\mathrm{d}\Gamma_Q, \\ B_5 &= \frac{p_z}{C}\int_\Gamma \frac{\partial}{\partial n_Q}[\nabla^2\tilde{L}_1 D_2(\Phi)]\,\mathrm{d}\Gamma_Q \end{aligned} \tag{26}$$

where the differential operators $\tilde{L}_1$ and $\tilde{L}_2$ are given in the Appendix, and

$$\begin{aligned} \tilde{L}'_8 &= (k_1+\nu k_2)D_1^2+[(2+\nu)k_1-k_2]D_2^2, \\ \tilde{L}'_9 &= (k_2+\nu k_1)D_2^2+[(2+\nu)k_2-k_1]D_1^2. \end{aligned} \tag{27}$$

Once the displacements and tractions are known over all the surface, the internal displacements can be computed using eqn (16). Substituting eqn (16) into eqn (3), one can calculate the internal stresses at any point. For example

$$\begin{aligned} N_x(p) = B\int_\Gamma \left\{\left[\frac{\partial T_{1j}}{\partial x_p}+\nu\frac{\partial T_{2j}}{\partial y_p}-(k_1+\nu k_2)T_{3j}\right]u_j-\left[\frac{\partial U_{1j}}{\partial x_p}+\nu\frac{\partial U_{2j}}{\partial y_p}-(k_1+\nu k_2)U_{3j}\right]t_j\right\}\mathrm{d}\Gamma_Q \\ +B\iint_\Omega \left[\frac{\partial U_{1j}}{\partial x_p}+\nu\frac{\partial U_{2j}}{\partial y_p}-(k_1+\nu k_2)U_{3j}\right]p_j\,\mathrm{d}\Omega_Q \quad (j = 1,2,\ldots,5) \end{aligned} \tag{28}$$

where $\partial/\partial x_p$ and $\partial/\partial y_p$ represent derivatives at the internal point p, and the formulations to solve other internal stress components can be deduced similarly.

NUMERICAL TREATMENTS AND EXAMPLES

According to the formulae obtained previously, the boundary of the shell is divided into a series of elements, and boundary displacements u_j and tractions t_j are approximated over each element through interpolation functions. Substituting these approximate functions into eqn (16), the discretized boundary integral equations are established for each nodal point of the boundary, and the integrals are computed over each boundary element. A system of linear algebraic equations involving the set of nodal tractions and nodal displacements is therefore obtained. For the theory of shallow shells involving shear deformation, there are five unknown tractions or displacements at each point of the boundary. Hence a system of $5m$ equations would be obtained after boundary conditions have been considered for m boundary nodal points on the surface. These equations can be solved by standard methods to obtain unknown boundary data, and the internal displacements and stresses can be obtained from eqns (16) and (28) respectively. According to the idea given above, a program has been written by the authors, in which linear interpolation functions are used, and some examples are given to check the formulae obtained.

(1) As shown in Fig. 2, a shallow spherical shell is simply supported on four sides under the action of uniform loading. The geometric and material parameters used are:

$$2a = 2b = 32\,\text{cm}, \quad \nu = 0.3, \quad E/p_z = 10^4, \quad k = \frac{1}{96}.$$

This example has been solved by Reddy (1984) for thick shell theory and the results of deflections at the middle point due to the variation in thickness h are listed in Table 1. We solved this problem by BEM, and the computed results are given for comparison. As symmetry can be treated automatically in the program designed by the authors, only a quarter of the shell boundary is taken into consideration and discretized in this example, and the number of discrete elements is represent by NN. It can be seen that the BEM results approach the results obtained by Reddy progressively as the element number NN increases.

(2) Consider a simply-supported elliptic parabolic shell under a uniformly distributed load. The geometric and material parameters are the same as in example 1 except that the curvature is now variable, that is $k_1 = 1/96$ and k_2 is variable. The computed results are listed in Table 2. The analytical and the numerical results obtained by other methods for comparison cannot be found. In the calculations, only a quarter of the boundary is divided into eight elements by considering symmetry. We can see from Table 2 that as the dimensionless parameter $\tau = k_2/k_1$ decreases, the deflection at the middle point of the shell

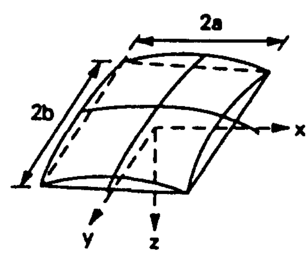

Fig. 2.

Table 1. Deflection w (at $x = 0$, $y = 0$) of a simply-supported shallow spherical shell under uniformly distributed load ($w \times 10^3$)

h		0.32	1.60	3.20	6.40
BEM	$NN = 2$	309.38	46.138	—	—
	$NN = 4$	—	43.438	10.867	—
	$NN = 8$	313.12	49.051	11.029	1.9311
Reddy (1984)		313.86	49.695	11.265	1.976

Table 2. Deflection w (at $x = 0$, $y = 0$) of a simply-supported elliptic parabolic shell under uniformly distributed load ($w \times 10^3$)

	h			
τ	0.32	1.60	3.20	6.40
1.0	313.12	49.051	11.029	1.9311
0.8	394.95	54.367	11.590	1.9650
0.5	587.46	64.576	12.408	2.0112
0.3	798.75	72.091	12.927	2.0402

increases gradually, and this increment tends to decelerate as the shell thickness, h, increases. This shows that spherical shell structures are more stable than variable curvature ones.

(3) Finally, a hyperbolic parabolic shell clamped on four sides is considered on which a uniform load is acting, as shown in Fig. 3, and the corresponding parameters are:

$$a = b = 20\,\mathrm{m}, \quad h = 0.2\,\mathrm{m}, \quad k_1 = -k_2 = \frac{1}{86}, \quad \nu = 0.$$

The BEM results are compared in Table 3 with the series results corresponding to classical thin shell theory obtained by He *et al.* (1981). It can be seen from this example that the treatments of the paper can also approximate thin shell results well when the thickness h is small.

CONCLUDING REMARKS

Based on the fundamental solution obtained by the authors, the boundary element treatments for shallow shells involving shear deformation were discussed in detail in this paper. The corresponding formulations were derived, and a program according to these formulae and treatments was written. Some numerical results were presented for several shells.

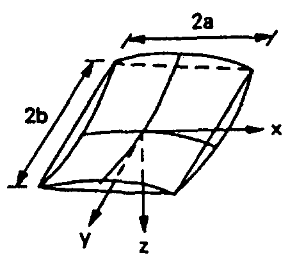

Fig. 3.

Table 3. Deflection and stress resultants of clamped hyperbolic parabolic shell under uniformly distributed load

Disp. & Str.			Coordinates		
			$x = 0$, $y = 0$	$x = a/2$, $y = 0$	$x = a$, $y = 0$
$w = \kappa_1\left(\frac{10p_z}{D}\right)$	κ_1	BEM	2.19	1.47	0.0
		He *et al.* (1981)	2.20	1.50	0.0
$M_x = \kappa_2 p_z$	κ_2	BEM	0.43	0.68	−2.32
		He *et al.* (1981)	0.48	0.60	−2.40
$N_x = \kappa_3(10p_z)$	κ_3	BEM	−4.21	−4.47	−5.28
		He *et al.* (1981)	−4.08	−4.25	−5.05

The work of this and the previous papers gives a new approach to solving this kind of shell problem. This is significant both in theory and application.

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APPENDIX

The elements of the symmetric differential operator matrix [F] and L are

$$L = \left[\nabla^8 - (1-\nu^2)\frac{B}{C}\nabla_k^4\bar{L}_2\right]\bar{L}_1,$$

$$F_{11} = \left[\frac{D}{B}\nabla^4\bar{L}_4 - \frac{D}{C}\bar{L}_2\bar{L}_6\right]\bar{L}_1,$$

$$F_{12} = \left[-\frac{D}{B}\frac{1+\nu}{1-\nu}\nabla^4 + \frac{D}{C}(k_1-k_2)^2\right]\bar{L}_1 D_1 D_2,$$

$$F_{13} = \frac{D}{C}\bar{L}_1\bar{L}_2\bar{L}_8, \quad F_{14} = \bar{L}_1\bar{L}_8 D_1, \quad F_{15} = \bar{L}_1\bar{L}_8 D_2,$$

$$F_{22} = \left[\frac{D}{B}\nabla^4\bar{L}_5 - \frac{D}{C}\bar{L}_2\bar{L}_7\right]\bar{L}_1,$$

$$F_{23} = \frac{D}{C}\bar{L}_1\bar{L}_2\bar{L}_9, \quad F_{24} = \bar{L}_1\bar{L}_9 D_1, \quad F_{25} = \bar{L}_1\bar{L}_9 D_2,$$

$$F_{33} = -\frac{D}{C}\nabla^4\bar{L}_1\bar{L}_2, \quad F_{34} = -\nabla^4\bar{L}_1 D_1, \quad F_{35} = -\nabla^4\bar{L}_1 D_2,$$

$$F_{44} = \bar{L}_3\left(\bar{L}_4 - \frac{2}{1-\nu}\frac{C}{D}\right) + \frac{2}{1-\nu}\frac{C}{D}\nabla^4 D_2^2,$$

$$F_{45} = -\frac{1+\nu}{1-\nu}\bar{L}_3 D_1 D_2 - \frac{2}{1-\nu}\frac{C}{D}\nabla^4 D_1 D_2,$$

$$F_{55} = \bar{L}_3\left(\bar{L}_5 - \frac{2}{1-\nu}\frac{C}{D}\right) + \frac{2}{1-\nu}\frac{C}{D}\nabla^4 D_1^2,$$

where,

$$D_1 = \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial y},$$

$$\bar{L}_1 = \nabla^2 - \frac{2}{1-\nu}\frac{C}{D}, \quad \bar{L}_2 = \nabla^2 - \frac{C}{D},$$

$$\bar{L}_3 = \nabla^6 - (1-\nu^2)\frac{B}{C}\nabla_k^4,$$

$$\bar{L}_4 = D_1^2 + \frac{2}{1-\nu}D_2^2, \quad \bar{L}_5 = D_2^2 + \frac{2}{1-\nu}D_1^2,$$

$$\bar{L}_6 = (k_1^2 + k_2^2 + 2\nu k_1 k_2)D_1^2 + 2(1+\nu)k_1^2 D_2^2,$$

$$\bar{L}_7 = (k_1^2 + k_2^2 + 2\nu k_1 k_2)D_2^2 + 2(1+\nu)k_2^2 D_1^2,$$

$$\bar{L}_8 = (k_1 + \nu k_2)D_1^3 + [(2+\nu)k_1 - k_2]D_1 D_2^2,$$

$$\bar{L}_9 = (k_2 + \nu k_1)D_2^3 + [(2+\nu)k_2 - k_1]D_1^2 D_2,$$

$$\nabla_k^2 = k_2\frac{\partial}{\partial x^2} + k_1\frac{\partial}{\partial y^2} = k_2 D_1^2 + k_1 D_2^2.$$

The elements of differential operator matrix [R] are

$$
\begin{aligned}
R_{1j} &= B[F_{1j}D_1 + vF_{2j}D_2 + (k_1 + vk_2)F_{3j}], \\
R_{2j} &= B[vF_{1j}D_1 + F_{2j}D_2 + (k_2 + vk_1)F_{3j}], \\
R_{3j} &= \tfrac{1}{2}(1 - v)B[F_{1j}D_2 + F_{2j}D_1], \\
R_{4j} &= C[F_{3j}D_1 + F_{4j}], \\
R_{5j} &= C[F_{3j}D_2 + F_{5j}], \\
R_{6j} &= D[F_{4j}D_1 + vF_{5j}D_2], \\
R_{7j} &= D[vF_{4j}D_1 + F_{5j}D_2], \\
R_{8j} &= \tfrac{1}{2}(1 - v)D[F_{4j}D_2 + F_{5j}D_1], \quad (j = 1, 2, \ldots, 5).
\end{aligned}
$$